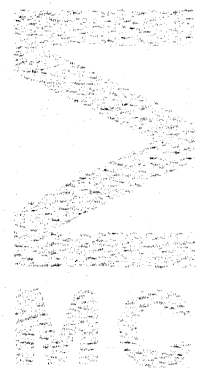


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AFDELING NUMERIEKE WISKUNDE  
(DEPARTMENT OF NUMERICAL MATHEMATICS)

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NOVEMBER

H.J.J. TE RIELE

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On generating new amicable pairs from given amicable pairs<sup>\*)</sup>

by

Herman J.J. te Riele

#### ABSTRACT

Rules are given for constructing new amicable pairs from given amicable pairs. By applying these rules to 1575 "mother" pairs known to the author, 1782 new amicable pairs were generated, so that the average "offspring" of these mother pairs is greater than 1. By a second, much more restricted application of the rules to the 400 smallest "daughter" pairs, 88 new "granddaughter" amicable pairs were generated.

KEY WORDS & PHRASES: *amicable pair*

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<sup>\*)</sup> This report will be submitted for publication elsewhere.

## 1. INTRODUCTION

An amicable pair is a pair of positive integers  $(m_1, m_2)$ ,  $m_1 \neq m_2$ , such that  $\sigma(m_1) = \sigma(m_2) = m_1 + m_2$ , where  $\sigma(\cdot)$  denotes the sum of divisors function. These number pairs have a long and interesting history (cf. [5]). Euler ([4]) was the first who systematically studied amicable pairs, and a great part of the known pairs were found with his methods and the use of electronic computers.

It is believed that there are infinitely many amicable pairs, although this has never been proved. In this note we present rules for finding new amicable pairs from given pairs. By applying these rules to a large list of known pairs, we have generated an average "offspring" of *more than one* "daughter" pair per given "mother" pair. We consider this result as strong numerical evidence for the existence of infinitely many amicable pairs.

In Section 2 we present one very simple rule, and two rules derived from it, by which we have generated 1782 new amicable pairs from a list of 1575 known mother pairs. These mother pairs were collected from lists of Lee and Madachy ([5], 1108 pairs), Costello ([3], 7 pairs), Borho ([1], 3 pairs), Woods ([10], 456 pairs) and te Riele ([7], 1 pair). By a second (more restricted, cf. Section 3) application of the rules to the 400 smallest pairs in the list of 1782 daughter pairs mentioned above, still 88 more new "granddaughter" amicable pairs were generated. All these 1870 new pairs are explicitly given in [8], including their prime factorizations. It should be noted that other rules for finding new amicable pairs from known pairs are already known (so-called Thabit-rules, see [1]), but up till now relatively few new pairs have actually been found with these rules (cf. [1,2,6]).

In Section 3 we give details of our computations, including some information about the mother pairs from which the largest numbers of daughter pairs were generated.

This work was inspired by the following observations. By inspecting the Lee & Madachy-list we found that there are certain pairs  $(m_1^{(i)}, m_2^{(i)})$ ,  $i = 1, 2, \dots, t$ ,  $t \geq 2$ , which share some common part, i.e., for which a pair of numbers  $(a_1, a_2)$  exists with  $a_1 > 1$  and  $a_2 > 1$  such that  $a_1$  is a divisor of  $m_1^{(i)}$  with  $\gcd(a_1, m_1^{(i)}/a_1) = 1$  and  $a_2$  is a divisor of  $m_2^{(i)}$  with  $\gcd(a_2, m_2^{(i)}/a_2) = 1$ , for  $i = 1, 2, \dots, t$ . For example, the five pairs (with

are numbered 1, 3, 4, 9 and 24, respectively, in [5]):

$$\begin{aligned} &\begin{cases} 220 = 2^2 5 \cdot 11 \\ 284 = 2^2 71 \end{cases} \begin{cases} 2620 = 2^2 5 \cdot 131 \\ 2924 = 2^2 17 \cdot 43 \end{cases} \begin{cases} 5020 = 2^2 5 \cdot 251 \\ 5564 = 2^2 13 \cdot 107 \end{cases} \\ &\begin{cases} 63020 = 2^2 5 \cdot 23 \cdot 137 \\ 76084 = 2^2 23 \cdot 827 \end{cases} \begin{cases} 308620 = 2^2 5 \cdot 13 \cdot 1187 \\ 389924 = 2^2 43 \cdot 2267 \end{cases} \end{aligned}$$

share the common part  $(a_1, a_2) = (2^2 5, 2^2)$ . Other examples are the five pairs numbered 6, 42, 90, 126 and 442 in [5] which share  $(2^3 17, 2^3 23)$  and the eight pairs numbered 8, 19, 20, 29, 121, 213, 260 and 282 in [5] which share  $(2^4 23, 2^4)$ .

*Notation.* We write  $\bar{n}$  for  $\sigma(n)$  and  $\tilde{n}$  for  $\sigma(n) - n$ .

## 2. THE RULES

RULE 1 (to find an amicable pair  $(a_1 p_1, a_2 q_1 q_2)$ )

*Find a solution  $(q_1, q_2)$  of the bilinear Diophantine equation*

$$(2.1) \quad D q_1 q_2 - \tilde{a}_1 \bar{a}_2 (q_1 + q_2) = a_1 \bar{a}_1 + \tilde{a}_1 \bar{a}_2, \quad D = a_1 a_2 - \tilde{a}_1 \tilde{a}_2,$$

for which both  $q_1$  and  $q_2$  are primes,  $q_1 \neq q_2$ , and  $\gcd(a_2, q_1 q_2) = 1$ . For such a solution, compute  $p_1$  from

$$(2.2) \quad \tilde{a}_1 p_1 = a_2 q_1 q_2 - \bar{a}_1.$$

If  $p_1$  is prime and  $\gcd(a_1, p_1) = 1$  then  $(a_1 p_1, a_2 q_1 q_2)$  is an amicable pair.  $\square$

In fact, this rule is well-known (cf. [5, p. 81, formulas (12)-(15)]), but the difficulty is to find numbers  $a_1$  and  $a_2$  for which  $q_1$ ,  $q_2$  and  $p_1$  are integral. Now we choose  $a_1$  and  $a_2$  as follows. Let  $(m_1, m_2)$  be a known amicable pair and write it as  $(a_1 p, a_2 q)$  where  $p$  and  $q$  are primes and  $\gcd(a_1, p) = \gcd(a_2, q) = 1$ . By using the definition of an amicable pair, one easily verifies that  $p$  and  $q$  can be expressed in terms of  $a_1$  and  $a_2$  as follows:

$$(2.3) \quad p = D^{-1}(\tilde{a}_2 \bar{a}_1 + a_2 \bar{a}_2), \quad q = D^{-1}(a_1 \bar{a}_1 + \tilde{a}_1 \bar{a}_2), \quad \text{where } D \text{ is defined in (2.1).}$$

So we have that  $D$  is a divisor of  $a_1\bar{a}_1 + \tilde{a}_1\bar{a}_2$ . Moreover, experiments have shown, that often  $D$  is also a divisor of  $\tilde{a}_1\bar{a}_2$ , or at least that  $D$  and  $\tilde{a}_1\bar{a}_2$  have a large common divisor. This means that for this choice of  $a_1$  and  $a_2$  the coefficient of  $q_1q_2$  in (2.1) can be made 1 (or at least a small integer  $> 1$ ) by dividing (2.1) by the greatest common divisor of  $D$ ,  $\tilde{a}_1\bar{a}_2$  and  $a_1\bar{a}_1 + \tilde{a}_1\bar{a}_2$ . This is a favorable situation for the existence of *integral* solutions  $q_1$  and  $q_2$ . Moreover, it turns out that often  $p_1$  defined by (2.2) is integral when  $q_1$  and  $q_2$  are integral.

EXAMPLE If we choose  $(a_1, a_2) = (2^2 5, 2^2)$  from the smallest known amicable pair (see Section 1), then we have  $D = 14$ ,  $\tilde{a}_1\bar{a}_2 = 154 = 11 \cdot 14$  and  $a_1\bar{a}_1 + \tilde{a}_1\bar{a}_2 = 994 = 71 \cdot 14$ . Eq. (2.1) reads  $q_1q_2 - 11(q_1+q_2) = 71$ , or, equivalently,  $(q_1-11)(q_2-11) = 192 = 2^6 3$ . The solutions  $(q_1, q_2) = (17, 43)$  and  $(13, 107)$  give  $p_1 = 131$  and  $p_1 = 251$ , respectively, so that with this rule we have generated the (known) pairs numbered 3 and 4 in [5] (cf. Section 1).  $\square$

The following two rules are generalizations of Rule 1, simply obtained by replacing  $a_1$  by  $a_1p_2$  (to give Rule 2) and  $a_2$  by  $a_2q_3$  (to give Rule 3). Here,  $p_2$  resp.  $q_3$  are "suitably" chosen primes (cf. (2.4) resp. (2.7) below).

RULE 2 (to find an amicable pair  $(a_1p_1p_2, a_2q_1q_2)$ )

Choose a prime  $p_2$  such that  $\gcd(a_1, p_2) = 1$  and

$$(2.4) \quad Dp_2 - \tilde{a}_2\bar{a}_1 > 0.$$

Find a solution  $(q_1, q_2)$  of the bilinear Diophantine equation

$$(2.5) \quad (Dp_2 - \tilde{a}_2\bar{a}_1)q_1q_2 - (\tilde{a}_1p_2 + \bar{a}_1)\bar{a}_2(q_1+q_2) = a_1\bar{a}_1p_2^2 + (a_1\bar{a}_1 + \tilde{a}_1\bar{a}_2)p_2 + \bar{a}_1\bar{a}_2$$

for which both  $q_1$  and  $q_2$  are primes,  $q_1 \neq q_2$ , and  $\gcd(a_2, q_1q_2) = 1$ . For such a solution compute  $p_1$  from

$$(2.6) \quad (\tilde{a}_1p_2 + \bar{a}_1)p_1 = a_2q_1q_2 - \bar{a}_1(p_2+1).$$

If  $p_1$  is prime,  $p_1 \neq p_2$ , and  $\gcd(a_1, p_1) = 1$  then  $(a_1p_1p_2, a_2q_1q_2)$  is an amicable pair.  $\square$

RULE 3 (to find an amicable pair  $(a_1 p_1, a_2 q_1 q_2 q_3)$ )

Choose a prime  $q_3$  such that  $\gcd(a_2, q_3) = 1$  and

$$(2.7) \quad Dq_3 - \tilde{a}_1 \bar{a}_2 > 0.$$

Find a solution  $(q_1, q_2)$  of the bilinear Diophantine equation

$$(2.8) \quad (Dq_3 - \tilde{a}_1 \bar{a}_2)q_1 q_2 - \tilde{a}_1 \bar{a}_2 (q_3 + 1)(q_1 + q_2) = a_1 \bar{a}_1 + \tilde{a}_1 \bar{a}_2 (q_3 + 1)$$

for which both  $q_1$  and  $q_2$  are primes,  $q_1 \neq q_2$ ,  $q_1 \neq q_3$ ,  $q_2 \neq q_3$ , and  $\gcd(a_2, q_1 q_2) = 1$ . For such a solution compute  $p_1$  from

$$(2.9) \quad \tilde{a}_1 p_1 = a_2 q_1 q_2 q_3 - \bar{a}_1.$$

If  $p_1$  is prime and  $\gcd(a_1, p_1) = 1$  then  $(a_1 p_1, a_2 q_1 q_2 q_3)$  is an amicable pair.  $\square$

Of crucial importance in Rules 2 and 3 is the choice of the primes  $p_2$  resp.  $q_3$ . As with (2.1),  $p_2$  and  $q_3$  should be chosen such that the coefficient of  $q_1 q_2$  in (2.5) resp. (2.8) is as small as possible (preferably 1) after dividing by the greatest common divisor of the three coefficients.

### 3. COMPUTATIONAL DETAILS

We have applied Rules 1-3 to the 1575 mother amicable pairs mentioned in the introduction. For each of these pairs  $(m_1, m_2)$  the following computations were carried out. We computed  $g := \gcd(m_1, m_2)$  and the quotients  $m_1/g$  and  $m_2/g$ . From these quotients all possible pairs of primes  $(p, q)$  were selected such that  $(m_1, m_2) = (a_1 p, a_2 q)$  with  $\gcd(a_1, p) = \gcd(a_2, q) = 1$ . To all the pairs  $(a_1, a_2)$  (and  $(a_2, a_1)$ ) obtained in this way, we applied Rules 1, 2 and 3 with the following restrictions. Let the bilinear equations (2.1), (2.5) and (2.8) be written as

$$(2.10) \quad c_1 q_1 q_2 - c_2 (q_1 + q_2) = c_3,$$

with  $\gcd(c_1, c_2, c_3) = 1$ . This is equivalent to

$$(2.10') \quad (c_1 q_1 - c_2)(c_1 q_2 - c_2) = c_1 c_3 + c_2^2 =: C.$$

In those cases where  $C$  was smaller than  $10^{25}$ ,  $C$  was completely factorized and for all possible products  $C = B_1 B_2$  the corresponding numbers  $q_i = c_1^{-1}(c_2 + B_i)$ ,  $i = 1, 2$ , were computed, checked on being integral, checked on primality, etc. In Rules 2 and 3 we chose for  $p_2$  resp.  $q_3$  successively the smallest ten primes satisfying (2.4) resp. (2.7), in order to retain  $c_1$  in (2.10) as small as possible.

In this way we have generated 1782 new amicable pairs from 1575 mother pairs, namely 170 with Rule 1, 1523 with Rule 2 and 89 with Rule 3. Of course, also (relatively few) *known* pairs were generated. The smallest new pairs found with Rules 1, 2 and 3 are, respectively:

$$\left\{ \begin{array}{l} 114944072 = 2^3 17 \cdot 19 \cdot 44483 \\ 125269528 = 2^3 53 \cdot 439 \cdot 673 \end{array} \right. \text{ with Rule 1 from } \left\{ \begin{array}{l} 726104 = 2^3 17 \cdot 19 \cdot 281 \\ 796696 = 2^3 53 \cdot 1879 \end{array} \right. ([5, \#37]),$$

$$\left\{ \begin{array}{l} 323401712 = 2^4 23 \cdot 431 \cdot 2039 \\ 332270608 = 2^4 101 \cdot 127 \cdot 1619 \end{array} \right. \text{ with Rule 2 from } \left\{ \begin{array}{l} 176272 = 2^4 23 \cdot 479 \\ 180848 = 2^4 89 \cdot 127 \end{array} \right. ([5, \#20]),$$

$$\left\{ \begin{array}{l} 1699746184 = 2^3 23 \cdot 41 \cdot 233 \cdot 967 \\ 1725115256 = 2^3 17 \cdot 12684671 \end{array} \right. \text{ with Rule 3 from } \left\{ \begin{array}{l} 10744 = 2^3 17 \cdot 79 \\ 10856 = 2^3 23 \cdot 59 \end{array} \right. ([5, \#6]).$$

The largest pair found is the pair of 38-digit numbers:

$$\left\{ \begin{array}{l} 84939420717490497547044056177577599145 = E \cdot 359 \cdot 40939 \cdot 44296620189660299 \\ 85084703583907612558249457600803200855 = E \cdot 911 \cdot 119883499 \cdot 5971269576509, \end{array} \right.$$

where  $E = 3^6 5 \cdot 19 \cdot 23 \cdot 137 \cdot 547 \cdot 1093$ , generated with Rule 2 from the pair  $(E \cdot 359 \cdot 144779, E \cdot 911 \cdot 57149)$ , which is pair #1083 of [5].

From 533 of the 1575 mother pairs we actually generated one or more new daughter pairs. The three "champion" mother pairs are:

$$\begin{cases} 77306245632044 = 2^2 11.13.47.6829.421079 \\ 85036870195156 = 2^2 11.1932656140799 \end{cases} \text{ which is pair \# 952 of [5],}$$

from which 85 new pairs were generated with Rule 2,

$$\begin{cases} 3693013664 = 2^5 41.131.21487 \\ 3812143072 = 2^5 119129471 \end{cases} \text{ which is pair \# 441 of [5],}$$

from which 37 new pairs were generated with Rule 2, and

$$\begin{cases} 76809600128 = 2^7 149.1151.3499 \\ 77414399872 = 2^7 604799999 \end{cases} \text{ which is pair \# 609 of [5],}$$

from which 31 new pairs were generated with Rule 2. We leave it to the reader to find out why in particular from *these* mother pairs so many daughter pairs could be generated. In [8] a frequency table is given consisting of all the mother pairs and the corresponding *numbers* of new daughter pairs generated from them with Rules 1, 2 and 3.

In a second step, we applied Rules 1, 2 and 3 to the 400 smallest of the 1782 daughter pairs, with the restriction that per daughter pair  $(m_1, m_2)$  we only considered one choice of  $(a_1, a_2)$ , viz., the pair obtained by dropping the largest prime divisor of  $m_1$  from  $m_1$  and the largest prime divisor of  $m_2$  from  $m_2$ . As in our first step, we also considered only those cases for which  $C < 10^{25}$  in  $(2.10')$ . This reduced the number of cases drastically, also because the average daughter pair considered was much larger in size than the average mother pair considered, which led to much larger values of  $C$  in  $(2.10')$ . Nevertheless, we found 88 new "granddaughter" pairs from the smallest 400 daughter pairs, by using Rules 1, 2 and 3. These are also given in [8] together with a frequency table, similar to the one described above.

Finally, we have listed in [8] a compressed numbered list of all the 1575 mother pairs, for referencing and checking purposes.

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